

The Loss of Unitarity in the Vicinity of a Time Machine

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We construct the propagator of a non-relativistic non-interacting particle in a flat spacetime in which two regions have been identified. This corresponds to the simplest “time machine”. We show that while completeness is lost in the vicinity of the time machine it holds before the time machine appears and it is recovered afterwards. Unitarity, however, is not satisfied anywhere. We discuss the implications of these results and their relationship to the loss of unitarity in black hole evaporation.

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Spacetime, and the phenomenon of gravitation, are described very well at a classical level by the theory of General Relativity. Locally, spacetime is isomorphic to Minkowski and there is a well defined lightcone and microscopic Causality. Globally however, things may be quite different. There is nothing in the laws of classical general relativity that prevents spacetimes from having closed causal (timelike or null) curves, that is future directed curves through a point p such that if one travels along them towards the future, one returns to the same spacetime point. It is easy to find examples of spacetimes in which closed timelike curves have always existed [1]. None of these examples, generally referred to as eternal time machines, look very much like our Universe. In each of these cases, it is not possible to pose the Cauchy problem for matter fields propagating in these spacetimes [2], and one can therefore believe that these spacetimes are rather pathological.

Another type of causality violation is one in which closed timelike curves develop during the evolution of a spacetime from some reasonable initial conditions. An example of such behavior is found in the Kerr solution which is believed to be the endpoint of gravitational collapse with rotation. The region in which causality violation occurs is close to the singularity and interior to the inner horizon. It might be the case that the Kerr example is generic under certain circumstances, as Tipler [3] has shown that if matter obeys the weak energy condition, and closed timelike lines develop to the future of some Cauchy surface, then the spacetime must be geodesically incomplete. If one believes in the cosmic censorship hypothesis [4], then under such circumstances, the singularity is always enclosed by a horizon, and we conjecture that if the weak energy condition is satisfied the closed timelike lines will also only occur in the interior of the horizon and the physics exterior to any horizon would always be unaffected by the paradoxes and difficulties associated with closed timelike curves. Hawking [5] has proposed the Chronology Protection Conjecture, presently still unproven, that would prevent causality violation under a wide range of circumstances.

Systems that obey the weak energy conditions classically, for example a free scalar field, do not necessarily obey it after quantization [6,7]. Under these circumstances, it appears to be possible to create a region of spacetime that includes closed timelike curves without

the occurrence of spacetime singularities, other than that associated with the chronology horizon. Similarly, since the laws of physics are time reversal invariant, we expect that such regions could disappear. This type of spacetime we refer to as a “time machine.” Morris, Thorne and Yurtsever [8] have shown one way that such spacetimes can arise, when a wormhole connects two spacelike separated points in Minkowski space. There are number of undesirable paradoxes that arise in such a spacetime. Recently several authors [9,10], discussed the resolution of these paradoxes within the realm of classical physics. However, in the presence of time machines the Cauchy problem fails to be well-posed in a very explicit way [9]. For each classical initial value problem, there exist an infinite number of consistent (i.e. non paradoxical) classical evolutions. In other words, although the paradoxes can be avoided, predictability will still be violated.

Our aim is to explore the nature of these quantum mechanical processes in the presence of time machines. In the absence of any microscopic quantum theory of gravity, we can only study quantum mechanical processes on a fixed spacetime background. Conceptually it is easiest to work in the Schrödinger picture. Then the state at time t , $|\psi(t)\rangle$, is determined in terms of the Hamiltonian operator, $H(t)$ and an initial state $|\psi(0)\rangle$. We cannot use this method here because the Hamiltonian only exists in spacetimes that are globally hyperbolic. Any spacetime that has closed timelike curves fails to be globally hyperbolic.

An alternative approach is to use the Feynman path integral to find the transition amplitude $\langle\psi(t)|\psi(0)\rangle$. The path integral can be derived from the Schrödinger formulation for certain classes of Hamiltonian, provided that the Hamiltonian exists [11]. However, in the absence of a Hamiltonian, the path integral is the only tool that there is, and we regards it as the fundamental definition.

We study a free non-relativistic particle. At a time t_i , the particle is in an eigenstate of position at x_i , so it is in the state $|i, t_i\rangle$. The propagator is the amplitude \mathcal{G}_{ji} given by:

$$\mathcal{G}_{ji} = \langle j, t_j | i, t_i \rangle = \sum \exp[iS_{ji}/\hbar] \quad (1)$$

where the summation is over all paths from (x_i, t_i) to (x_j, t_j) and S_{ji} is the classical action

evaluated along the path in question.

According to the postulates of quantum mechanics the propagator must obey the group properties of completeness:

$$\mathcal{G}_{ji} = \sum_k \mathcal{G}_{jk} \mathcal{G}_{ki} \quad t_j \geq t_k \geq t_i \quad (2)$$

and unitarity:

$$\sum_k \mathcal{G}_{ki}^* \mathcal{G}_{kj} = \begin{cases} \delta_{ij}, & \text{if } t_i = t_j < t_k \\ \mathcal{G}_{ji}, & \text{if } t_i < t_j < t_k \\ \mathcal{G}_{ij}^*, & \text{if } t_j < t_i < t_k \end{cases} . \quad (3)$$

Completeness asserts that if one examines \mathcal{G}_{ik} , then the particle will have been at some position at any intermediate time t_j . Unitarity is the statement that it is possible to reverse the time evolution of a system so as to reconstruct an earlier state of the system given the state at a later instant of time. Unitarity can be viewed as being equivalent to conservation of probability. It should be noted that completeness and unitarity ensure that the time evolution of a system is described by elements of a group, since the additional axiom of associativity is clearly satisfied as a consequence of (2). If the Hamiltonian exists, and it is Hermitian, then completeness and unitarity are trivially satisfied, as H is a generator of the Lie algebra associated with the group of time evolution.

As an explicit example consider the propagator K_{ji} of a free non relativistic particle of mass m propagating in a flat spacetime [11].

$$K_{ji} = \begin{cases} \left(\frac{m}{2\pi i \hbar (t_j - t_i)} \right)^{\frac{3}{2}} \exp \left(\frac{im(x_j - x_i)^2}{2\hbar(t_j - t_i)} \right), & \text{if } t_j > t_i \\ 0, & \text{if } t_j < t_i \end{cases} . \quad (4)$$

K_{ji} vanishes if $t_i > t_j$ since non relativistic particle propagates only to the future. Since we are dealing with a non relativistic propagator, the light-cone is the line $t = \text{const}$, that is a particle located at (x_i, t_i) can propagate to any point in which $t > t_i$. Clearly K_{ji} obeys (2) and (3), and can be derived by either Hamiltonian methods or by path integrals.

Now we wish to study what happens quantum mechanically to particles traveling in a spacetime that has closed timelike lines. Firstly, we need to find a model spacetime in which calculations are straightforward but nevertheless has the properties of a realistic time machine. We therefore consider a flat spacetime in which we identify two selected spatial regions V_- and V_+ in such a way that for each point $(x_-, t_-) \in V_-$ we match another point $(x_+, t_+) \in V_+$ with $t_+ > t_-$. This identification mimics, in a simple way the effect of connecting two timelike separated regions by a wormhole and hence, by convention we refer to the connection between (x_+, t_+) and (x_-, t_-) as the “wormhole.” The identification is so that a future-directed timelike line arriving at (x_+, t_+) emerges at (x_-, t_-) again traveling toward the future and a future-directed timelike line arriving at (x_-, t_-) will emerge at (x_+, t_+) . We call this a two sided wormhole [12]. We denote by W_{-+} and W_{+-} the propagators “inside” the wormhole. Since we just identify the points in V_+ and V_- , W_{-+} and W_{+-} degenerate to an identity function.

Had there been no wormhole then the propagator in the time-machine spacetime, \mathcal{G}_{ji} , would simply become K_{ji} given by equation (4). However, it is not too hard to evaluate (1) explicitly in the time-machine spacetime which we are considering, because it is simple to find all possible paths by which a particle can propagate from (x_i, t_i) to (x_j, t_j) . We will explicitly calculate \mathcal{G}_{ji} for the case that $t_i < t_- < t_+ < t_j$. The possible paths are then labeled by the number of times n that the particle traverses the wormhole, and the contribution to \mathcal{G}_{ji} from all paths with fixed n is $\mathcal{G}_{ji}^{(n)}$. For $n = 0$ we have:

$$\begin{aligned} \mathcal{G}_{ji}^{(0)} = & K_{ji} - \int_+ d^3x K_{j+} K_{+i} - \int_- d^3x K_{j-} K_{-i} \\ & + \int_- d^3x \int_+ d^3x' K_{j+'} K_{+'-i} K_{-i} \end{aligned} \quad (5)$$

where K_{ji} is the ordinary propagator in the flat spacetime given by (4) and \int_+ (\int_-) denotes integration over the volume V_+ (V_-). The first term comes from all paths that go from (x_i, t_i) to (x_j, t_j) . However the second and third terms, which represent the contribution of all paths from (x_i, t_i) to (x_j, t_j) via V_+ , and all paths from (x_i, t_i) to (x_j, t_j) via V_- respectively, must be subtracted off since any particle that arrives at (x_+, t_+) has traveled via the wormhole and

emerged at (x_-, t_-) (and similarly for particles arriving at (x_-, t_-)). In these subtractions we have double counted the paths that in ordinary space would have gone from (x_i, t_i) via V_- to V_+ and then to (x_j, t_j) and those are added in the last term.

For $n = 1$ the calculation can be done in much the same way:

$$\begin{aligned}\mathcal{G}_{ji}^{(1)} &= \int_+ d^3x \int_- d^3x' \left[K_{j-'} - \int_+ d^3x'' K_{j+''} K_{+''-'} \right] \\ &\quad W_{-'+} K_{+i} + \int_- d^3x \int_+ d^3x' K_{j+'} W_{+'-} K_{-i} \\ &= K_{j-} W_{-+} K_{+i} - K_{j+'} K_{+''-} W_{-+} K_{+i} + K_{j+} W_{+-} K_{-i}\end{aligned}\tag{6}$$

where in the second equality we adopt the notation, like the summation convention, that repeated *adjacent* indices are integrated over (thus the indices behave like matrix indices). The first term in (6) is the contribution from a particle traveling through the wormhole once. The second represents paths that traverse the wormhole once and then would have traveled to (x_j, t_j) via (x_+, t_+) . Such paths are doomed to travel through the wormhole once more and those will contribute to $\mathcal{G}_{ji}^{(n)}$ with $n \geq 2$. The last term represents the contribution of paths that reached (x_-, t_-) and have traveled through the wormhole to (x_+, t_+) and from there to (x_j, t_j) .

Similarly one can construct the general $\mathcal{G}_{ji}^{(n)}$ for paths traversing the wormhole n times. Unlike the previous two cases for $n \geq 2$ we have contributions only from paths that began by reaching (x_+, t_+) :

$$\begin{aligned}\mathcal{G}_{ji}^{(n)} &= K_{j-} W_{-+} \underbrace{K_{+-'} W_{-'+} K_{+''-''} W_{-''+''} \dots K_{+'''-'''} W_{-'''+'''}}_{(n-1) \text{ times}} K_{+''''i} \\ &\quad - K_{j+} \underbrace{K_{+-'} W_{-'+} K_{+''-''} W_{-''+''} \dots K_{+'''-'''} W_{-'''+'''}}_{n \text{ times}} K_{+''''i}\end{aligned}\tag{7}$$

The complete propagator can now be evaluated in terms of K_{ji} by:

$$\begin{aligned}\mathcal{G}_{ji} &= \sum_{n=0}^{\infty} \mathcal{G}_{ji}^{(n)} = K_{ji} + (K_{j-} W_{-+} - K_{j+})(\delta_{++'} - K_{+-'} W_{-'+})^{-1} K_{+i} \\ &\quad + (K_{j+} W_{-+} - K_{j-} + K_{j+} K_{+-}) K_{-i}.\end{aligned}\tag{8}$$

Note that $\delta_{++'}$ has the dimension of L^{-3} . $(\delta_{++'} - K_{+-'} W_{-'+})^{-1}$ is to be regarded as a matrix inverse. The propagator \mathcal{G}_{ji} was derived assuming that $t_i < t_-$ and that $t_j > t_+$. However,

(8) holds for all possible time orderings provided one recalls from (4) that $K_{ji} = 0$ if $t_j < t_i$, as can readily be seen by considering all of possible time ordering of t_i , t_- , t_+ and t_j . The last term does not appear if we choose a one sided wormhole, instead of the two sided one that we are considering [12].

To simplify the calculations we consider now V_+ and V_- of spatial extent $\Delta x \ll \hbar T/mX$ where T and X are any of the time or length scales in the problem (e.g. $t_+ - t_- \dots$). Then, each of the propagators can be taken to be constant over V_+ and V_- . If v is the volume of V_+ (and V_-), then we find that [13]:

$$\begin{aligned} \mathcal{G}_{ji} \approx & K_{ji} + v \frac{(K_{j-} - K_{j+})K_{+i}}{1 - vK_{+-}} \\ & + v(K_{j+} - K_{j-} + vK_{j+}K_{+-})K_{-i} \end{aligned} \quad (9)$$

where we used the fact W is an identity function and thus $K_{j\pm}W_{\pm\mp} \approx K_{j\mp}$.

To examine the completeness properties of \mathcal{G}_{ij} we need to evaluate $\sum_k \mathcal{G}_{jk}\mathcal{G}_{ki}$ for the various cases of t_j , t_k and t_i greater or less than t_- and t_+ and subject to $t_i < t_k < t_j$. We use the completeness properties of K_{ij} , (2), together with $\sum_k K_{+k}K_{k+} = 0$. The latter follows from the fact that either K_{+k} or K_{k+} vanishes depending on whether t_k is greater then or less then t_+ . If $t_k > t_+$ or $t_k < t_-$, completeness of \mathcal{G}_{ij} follows directly from the completeness of K_{ij} . Hence, \mathcal{G}_{ij} obeys the completeness condition if the intermediate surface ($t = t_k$) is chosen to be either to the past or the future of the time machine. If however $t_- < t_k < t_+$ then

$$\begin{aligned} \sum_k \mathcal{G}_{jk}\mathcal{G}_{ki} = & K_{ji} + v \left(\frac{K_{+-}K_{j+} - K_{j-}}{1 - vK_{+-}} \right) K_{-i} \\ & + v \left(\frac{K_{j-}(2 - vK_{+-}) - K_{j+}}{(1 - vK_{+-})^2} \right) K_{+i} \end{aligned} \quad (10)$$

and completeness fails to be satisfied. This is because the particle can cross such an intermediate surface exterior to the wormhole any number of times. This violation of completeness seems harmless as it happens only while the time-machine is operating and completeness is recovered after the time-machine has ceased to exist. Indeed had we considered a surface

intersecting the interior of the wormhole we would have discovered no violation of completeness.

If we try to check unitarity either by using the unitarity of K_{ji} , or by the explicit functional form of K_{ji} , (4), we discover that the unitarity condition is violated, unless all t_i , t_j and t_k are to either the past or to the future of the time machine, in which case $\mathcal{G}_{ij} = K_{ij}$. As an example of violation of unitarity we consider the special case of: $t_i = t_j < t_- < t_+ < t_k$, i.e. the initial and final points are to the past of the time machine and the intermediate point is to the future of it (it will be too lengthy to write out here all possible orderings of t_i, t_j, t_k, t_- and t_+). Then

$$\begin{aligned} \sum_k \mathcal{G}_{kj}^* \mathcal{G}_{ki} &= \delta_{ji} + v \left(\frac{K_{+i}}{1 - vK_{+-}} \right) \left(-K_{+j} + K_{+j}^* + K_{-j} - K_{-j}^* + v^2 K_{+-}^* K_{-j}^* \right) \\ &\quad + v \left(\frac{K_{+j}^*}{1 - vK_{+-}^*} - K_{-j}^* \right) \left(-K_{+i}^* + K_{+i} - K_{-i} + K_{-i}^* + v^2 K_{+-}^2 K_{-i} \right) \\ &\quad + v^2 K_{+-}^* K_{-j}^* \left[v^2 K_{+-} K_{-i} + (1 - vK_{+-}^*) K_{-i} + K_{+i}^* \right] + v^2 K_{+j} K_{+-} K_{-i} \end{aligned} \quad (11)$$

and unitarity is broken [13]. The physical reason for unitarity violation is that the path integral for the reverse process is not given simply by the complex conjugate of \mathcal{G}_{ji} as is usually the case. In fact it is not possible to construct an inverse to \mathcal{G}_{ij} as can be seen by attempting to construct the inverse of \mathcal{G}_{ij} order by order in n .

We have shown that in this simplified model for a time machine unitarity is violated, although completeness is not. This means that there is no longer a unitary time evolution operator for this system, and so the canonical formulation of quantum mechanics is inapplicable. However, it should be noted that despite the fact that the group property of time evolution is violated, the fact that completeness is preserved means that time evolution can be described by a semigroup [15]. This means that one can predict the future from a given microscopic theory given data specified before the time machine was formed, but it is impossible to reconstruct the past on the basis of what one can describe to the future of a time machine. This is the quantum analog of the results of Echeverria, Thorne and Klinkhammer [9].

From this simple picture, it is clear that quantum mechanics (as usually formulated) breaks down in such spacetimes essentially because of the existence of a closed timelike curves. If we try to extend our calculations to more complicated or realistic cases- for example by having relativistic particles- exactly the same phenomenon will occur because the pathologies are associated with the non-existence of a Hamiltonian due to the appearance of close timelike lines. However, in these cases, it will be more complicated to see explicitly how the difficulties arise.

The breakdown of quantum mechanics discussed here is very reminiscent of the phenomenon of black hole evaporation [7]. If black holes evaporate completely, then it seems likely that information about the collapsing matter is annihilated, as a consequence of lack of a unitary time evolution which manifest itself as a pure state developing into a mixed state.

We would like to conclude with some speculations. Standard Hamiltonian quantum mechanics is violated by closed timelike curves, so we can suppose that if the laws of nature are truly quantum mechanical then it will be impossible to construct time machines. If however, as has been suggested on the basis of black hole physics, quantum mechanics breaks down when gravitation is taken into account [16] (i.e. black hole evaporation when non-unitary evolution of similar nature also appears to take place) then we see no reason why it should not be possible to construct such machines [17].

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placed back in the spacetime there, but a particle that reaches (x_-, t_-) is not transported inside the wormhole to (x_+, t_+) . Our main result holds also for the one-way wormhole.

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